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On General Differentiation.

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1. Assuming that

$$D^n x^m = \frac{f(m+1)}{f(m-n+1)} x^{m-n}, \tag{1}$$

we see at once (on writing $n = 1$) that

$$f(m+1) = mf(m). \tag{2}$$

This result is given by nearly every writer on the subject, most of whom assume some particular value for $f(m)$. Of these assumptions that of Peacock is the simplest, viz. $f(m) = \Gamma(m)$, but Kelland shows that (2) is satisfied by other functions besides Gamma-Functions (Trans. Roy. Soc. Edinb., Vol. XIV. p. 578).

In fact, the general solution of (2) considered as a difference-equation is

$$f(m) = C_m \Gamma(m), \tag{3}$$

where $C_{m+1} = C_m$, and it may be noticed that Liouville's formula,

$$D^n x^m = (-)^n \frac{\Gamma(-m+n)}{\Gamma(-m)} x^{m-n},$$

corresponds to the particular solution

$$f(m) = (-)^{m-1} \frac{\sin m\pi}{\pi} \Gamma(m) = (-)^{m-1} \frac{1}{\Gamma(1-m)},$$

and that (3) may also be written in the form

$$f(m) = (-)^{m-1} \frac{C'_m}{\Gamma(1-m)}, \tag{4}$$

where $C'_{m+1} = C'_m$.

Now, A being any finite quantity which does not contain x, we have from (1),

$$D^n \Sigma A \frac{f(m-n+1)}{f(m+1)} x^m = \Sigma A x^{m-n},$$

and in the cases in which $\dfrac{f(m+1)}{f(m-n+1)} = \infty$,

$$D^n \,.\, 0 = \Sigma A x^{m-n}, \tag{5}$$

where the only conditions are that A is finite and independent of x, and that the summation extends to all values of m for which $\dfrac{f(m+1)}{f(m-n+1)}$ is infinite.

When n is a positive integer, the value of this fraction is, by (2),

$$m(m-1)\ldots.(m-n+1),$$

a finite quantity; so that in this case (5) contains no terms, or there is no complementary function.

But when n is a negative integer $= -p$, the fraction is, by (3),

$$\frac{f(m+1)}{f(m+p+1)} = \frac{C_{m+1}\Gamma(m+1)}{C_{m+p+1}\Gamma(m+p+1)} = \frac{\Gamma(m+1)}{\Gamma(m+p+1)}$$

and is infinite for the values $-1, -2, -3, \ldots. -p$ of m, and for no others; so that in this case (5) becomes

$$D^{-p} \,.\, 0 = A_1 x^{p-1} + A_2 x^{p-2} + \ldots. + A_p.$$

Hence in every system of general differentiation we have, when n is a positive integer,

$$D^n \,.\, 0 = 0, \text{ and } D^{-n} \,.\, 0 = (1_1 x)^{n-1}.$$

The value of the complementary function may be obtained in a more definite form in each particular case; for example, if $f(m) = \Gamma(m)$ (Peacock's value), we have

$$\frac{A_1 x^{-1}}{\Gamma(0)} + \frac{A_2 x^{-2}}{\Gamma(-1)} + \frac{A\; x_{-3}}{\Gamma(-2)} + \ldots. = 0;$$

and, operating with D^n,

$$D^n \,.\, 0 = \frac{A_1 x^{-1-n}}{\Gamma(-n)} + \frac{A_2 x^{-2-n}}{\Gamma(-1-n)} + \frac{A_3 x^{-3-n}}{\Gamma(-2-n)} + \ldots., \tag{6}$$

we obtain a form of the complementary function which satisfies the required conditions. This result is the same as (5), for $\dfrac{\Gamma(m+1)}{\Gamma(m-n+1)}$ can only become infinite for negative integral values of m.

Using Liouville's value of $f(m)$, the fraction $\dfrac{f(m+1)}{f(m-n+1)}$ becomes $(-)^n \dfrac{\Gamma(-m+n)}{\Gamma(-m)}$, which is only infinite for positive integral values of $m-n$, so that (5) contains in this case positive integral powers of x only.

Here again, since $f(m) = 0$ for positive integral values of m,

$$A_0 f(1)\, x^n + A_1 f(2)\, x^{n+1} + A_2 f(3)\, x^{n+2} + \ldots. = 0;$$

operating, as before, with D^n, we get

$$D^n \,.\, 0 = A_0 f(n+1) + A_1 f(n+2)\, x + A_2 f(n+3)\, x^2 + \ldots., \tag{7}$$

a form of the complementary function such that, when n is a positive integer,

$$D^n . 0 = 0, \text{ and } D^{-n} . 0 = (1_1 x)^{n-1}.$$

As an additional example consider the case $f(m) = \cos 2m\pi\Gamma(m)$. Here $\frac{\cos 2m\pi\Gamma(m+1)}{\cos 2(m-n)\pi\Gamma(m-n+1)}$ is infinite (1) when m is a negative integer, and (2) when $(m-n)$ is a fraction with an odd numerator, and with denominator 4; so that the complementary function is, by (5),

$$A_1 x^{-1-n} + A_2 x^{-2-n} + \dots . B_1 x^{\frac{1}{4}} + B_2 x^{\frac{3}{4}} + \dots . + C_1 x^{-\frac{1}{4}} + C_2 x^{-\frac{3}{4}} + \dots .$$

The first part of this must be of the form (6), and it is easy to show that the second and third parts each contain a factor sin $n\pi$.

2. Whatever system of general differentiation we use, we see from (1) and (5) that whenever a term of the differentiated expression is similar to a term of the complementary function, its coefficient is infinite, and a correction must be used which will introduce logarithmic terms; in fact, when $\frac{f(m+1)}{f(m-n+1)}$ is infinite and h small,

$$D^n x^{m+h} = \frac{f(m+h+1)}{f(m+h-n+1)}\left(x^{m+h-n} - x^{m-n}\right),$$

the latter term being part of the complementary function,

$$= \frac{f(m+h+1)\, x^{m-n}}{f(m+h-n+1)} . h \log x \qquad \text{(in the limit).}$$

Or,

$$D^n x^m = x^{m-n} \log x \left[\frac{f(m+h+1)\, h}{f(m+h-n+1)}\right]_{h=0}.$$

In the first, or Peacock's case, putting $m = -\mu$ and $f = \Gamma$, this becomes, after some easy reductions,

$$D^n x^{-\mu} = \frac{(-)^{\mu-1} x^{-\mu-n} \log x}{(\mu-1)!\, \Gamma(-\mu-n+1)}. \tag{8}$$

Or, what is the same thing,

$$D^n x^{-\mu} = -\frac{\sin n\pi\Gamma(\mu+n)\, x^{-\mu-n} \log x}{\pi(\mu-1)!}. \tag{9}$$

In the second, or Liouville's case, putting $m - n = \mu$, we have

$$D^n x^{n+\mu} = \frac{f(n+\mu+1)}{\mu!\, f(0)} x^\mu \log x, \tag{10}$$

where in all three formulæ μ is and n is not a positive integer, and the complementary function is omitted; and in (10) $f(m)$ has either of the equivalent forms $(-)^{m-1} \frac{\sin m\pi\Gamma(m)}{\pi}$ or $(-)^{m-1} \frac{1}{\Gamma(1-m)}$.

If now, as a verification, n is put $= -\mu$ in (8) and $= -\mu - 1$ in (10), the results are

$$D^{-\mu}x^{-\mu} = \frac{(-)^{\mu-1}}{(\mu-1)!}\log x, \qquad \text{and } D^{-\mu-1}x^{-1} = \frac{x^{\mu}}{\mu!}\log x.$$

From the particular case $\mu = 0$ in (10) it is easy to deduce the values of $D^n \log x$ and $D^{-n}\log x$, which are found to be the same as those given by Greatheed (Camb. Math. Journ., Vol. I.) and Kelland (Trans. Roy. Soc. Edinb., Vol. XIV. p. 588), viz.

$$\left.\begin{aligned} D^{n}\ \log x &= (-)^{n-1}\,\Gamma(n)\,x^{-n} \\ D^{-n}\log x &= (-)^{n}\,\frac{\pi}{\sin n\pi\Gamma(1+n)}\,x^{n} \end{aligned}\right\}. \tag{11}$$

3. But a more general value of $D^n \log x$ may be found as follows:

Since
$$D^{n}\,\frac{x^{h}-1}{h} = \frac{f(h+1)\,x^{h-n}}{hf(h-n+1)} - \frac{f(1)\,x^{-n}}{hf(1-n)} + D^{n}.\,0,$$

we have, taking the limit when $h = 0$,

$$D^{n}\log x = x^{-n}\left[\frac{f(h+1)\,x^{h}}{hf(h-n+1)} - \frac{f(1)}{hf(1-n)}\right]_{h=0} + D^{n}.\,0.$$

Whence

$$D^{n}\log x = \frac{f(1)}{f(1-n)}\,x^{-n}\log x + \lambda_{n}x^{-n} + D^{n}.\,0, \tag{12}$$

λ_n being written for $\qquad \left[\dfrac{f(h+1)}{hf(h-n+1)} - \dfrac{f(1)}{hf(1-n)}\right]_{h=0}.$

The value of λ_n may be found in various ways; in Liouville's case, $f(1) = 0$, and the value of λ_n is by (11) $(-)^{n-1}\Gamma(n)$, unless n be a negative integer; in Peacock's case, we have

$$D^{n}\log x = \frac{x^{-n}\log x}{\Gamma(1-n)} + \lambda_{n}x^{-n} + D^{n}.\,0. \tag{13}$$

and

$$\begin{aligned}\lambda_{n} &= \left[\frac{\Gamma(h+1)}{h\Gamma(h-n+1)} - \frac{\Gamma(1)}{h\Gamma(1-n)}\right]_{h=0} = \frac{1}{\Gamma(-n)}\int_{0}^{1}\left[(1-x)^{-n-1}\,\frac{x^{h}-1}{h}\right]_{h=0} dx \\ &= \frac{1}{\Gamma(-n)}\int_{0}^{1}\frac{\log x\,dx}{(1-x)^{n+1}},\end{aligned}$$

and this determination is unsuitable when n is a positive integer.

The value of λ_n is easily seen to be $\dfrac{f'(1)\,f(1-n) - f(1)\,f'(1-n)}{\{f(1-n)\}^{2}}$, or it may be found from a difference-equation, as follows:

Writing (12) in the form

$$x^{n}D^{n}\log x = \frac{f(1)}{f(1-n)}\log x + \lambda_{n} + x^{n}D^{n}.\,0,$$

and remembering that

$$(xD - n)\,x^{n}D^{n} = x^{n+1}D^{n+1},$$

we have

$$x^{n+1}D^{n+1}\log x = (xD - n)\left\{\frac{f(1)}{f(1-n)}\log x + \lambda_{n}\right\} + x^{n+1}D^{n+1}.\,0.$$

Therefore

$$\frac{f(1)}{f(-n)}\log x + \lambda_{n+1} = (xD - n)\left\{\frac{f(1)}{f(1-n)}\log x + \lambda_{n}\right\};$$

or, since $f(1-n) = -nf(-n)$,

$$\lambda_{n+1} + n\lambda_{n} = \frac{f(1)}{f(1-n)}\,. \tag{14}$$

When n is any integer the functions f reduce to Gamma-Functions, so that, if n be positive and integral,

$$\left.\begin{aligned}\lambda_{n+1} + n\lambda_{n} &= 0\\ \lambda_{-n+1} - n\lambda_{-n} &= \frac{1}{n\,!}\end{aligned}\right\} \tag{15}$$

are the equations which respectively determine the coefficient of x^{-n} in the n^{th} differential and that of x^{n} in the n^{th} integral of $\log x$.

4. We cannot in general assume that $D^{n}e^{ax} = a^{n}e^{ax}$, but the complementary function can always be determined, so that

$$D^{n+1}e^{ax} = aD^{n}e^{ax}, \tag{16}$$

and with this value of the complementary function

$$D^{n}e^{ax} = C_{n}\,a^{n}e^{ax},$$

where C_{n} is any function of n subject to the conditions $C_{n+1} = C_{n}$ and $C_{0} = 1$. For example, in Peacock's case,

$$D^{n}e^{ax} = \frac{x^{-n}}{\Gamma(1-n)} + \frac{ax^{1-n}}{\Gamma(2-n)} + \frac{a^{2}x^{2-n}}{\Gamma(3-n)} + \cdots\cdot + D^{n}.\,0, \tag{17}$$

the proper determination is

$$D^{n}.\,0 = \frac{a^{-1}x^{-1-n}}{\Gamma(-n)} + \frac{a^{-2}x^{-2-n}}{\Gamma(-1-n)} + \frac{a^{-3}x^{-3-n}}{\Gamma(-2-n)} + \cdots\cdot;$$

and substituting this value of $D^{n}.\,0$ in (17), we get an expression for $D^{n}e^{ax}$ which satisfies (16) and reduces to $a^{n}e^{ax}$ when n is either a positive or a negative integer.

From (17), omitting the complementary function and changing the sign of n, we deduce

$$D^{-n}e^{x} = \frac{x^{n}}{\Gamma(n+1)} + \frac{x^{n+1}}{\Gamma(n+2)} + \frac{x^{n+2}}{\Gamma(n+3)} + \cdots\cdot, \tag{18}$$

a series that may be summed as follows:

Let S denote the sum of the series, then

$$\frac{dS}{dx} = \frac{x^{n-1}}{\Gamma(n)} + S.$$

Solving this equation in the usual manner, we obtain

$$S = \frac{e^x}{\Gamma(n)} \int_0^x e^{-x} x^{n-1} dx,$$

the lower limit being determined from the circumstance that, n being positive, S vanishes with x.

Hence, restoring the complementary function, we have, when n is positive,

$$D^{-n} e^x = \frac{e^x}{\Gamma(n)} \int_0^x e^{-x} x^{n-1} dx + D^{-n} . 0. \tag{19}$$

5. Expanding $(x + y)^m$ on the supposition $y < x$, and omitting the complementary function, we have

$$D_x^n (x + y)^m = D_x^n \left(x^m + m x^{m-1} y + \frac{m(m-1)}{1.2} x^{m-2} y^2 + \dots \right)$$

$$= \frac{f(m+1)}{f(m-n+1)} x^{m-n} + \frac{m f(m)}{f(m-n)} x^{m-n-1} y + \frac{m(m-1) f(m-1)}{1.2 f(m-n-1)} x^{m-n-2} y^2 + \dots$$

$$= \frac{f(m+1)}{f(m-n+1)} \left\{ x^{m-n} + (m-n) x^{m-n-1} y + \frac{(m-n)(m-n-1)}{1.2} x^{m-n-2} y^2 + \dots \right\}$$

$$= \frac{f(m+1)}{f(m-n+1)} (x + y)^{m-n} \tag{20}$$

$$D_y^n (x + y)^m = D_y^n \left(x^m + m x^{m-1} y + \frac{m(m-1)}{1.2} x^{m-2} y^2 + \dots \right)$$

$$= \frac{f(1)}{f(1-n)} x^m y^{-n} + \frac{m f(2)}{f(2-n)} x^{m-1} y^{1-n} + \frac{m(m-1) f(3)}{1.2 . f(3-n)} x^{m-2} y^{2-n} + \dots$$

$$= f(1) \left\{ \frac{x^m y^{-n}}{f(1-n)} + \frac{m x^{m-1} y^{1-n}}{f(2-n)} + \frac{m(m-1) x^{m-2} y^{2-n}}{f(3-n)} + \dots \right\} \tag{21}$$

When n is a positive integer, (21) coincides with (20), which may be considered as the principal form of $D^n (x + y)^m$; when n is a negative integer, (21) becomes

$$\frac{x^m y^n}{n!} + m x^{m-1} \frac{y^{n+1}}{(n+1)!} + m(m-1) x^{m-2} \frac{y^{n+2}}{(n+2)!} + \dots$$

which is a form of the n^{th} integral of $(x + y)^m$ with respect to y, and again the two forms will be equivalent if the complementary function be properly chosen.

When n is fractional, (21) vanishes in Liouville's case, since $f(1) = 0$, but then (20), which contains only positive integral powers of y, forms part of $D_y^n . 0$.

But in Peacock's case $D_y^n . 0$ contains only negative fractional powers of y, so that no determination of the complementary function can make the two forms coincide. Hence we see that in general, when n is fractional, we cannot assume that

$$D_n^x \phi(x+y) = D_y^n \phi(x+y).$$

The two forms of $D^n(1+x)^m$ obtained from (20) and (21) are,

$x > 1$

$$D^n(1+x)^m = \frac{f(m+1)}{f(m-n+1)}(1+x)^{m-n} + D^n . 0, \tag{22}$$

$x < 1$

$$D^n(1+x)^m = f(1)\left\{\frac{x^{-n}}{f(1-n)} + \frac{mx^{1-n}}{f(2-m)} + \frac{m(m-1)x^{2-n}}{f(3-n)} + \dots\right\} + D^n.0; \tag{23}$$

and to these may be added two other formulæ obtained from (22) by considering the cases of failure as in (8) and (10), viz.

$x > 1$

$$D^n(1+x)^{-\mu} = \frac{(-)^{\mu-1}(1+x)^{-\mu-n}\log(1+x)}{(\mu-1)!\,\Gamma(-\mu-n+1)} + D^n . 0, \tag{24}$$

$$D^n(1+x)^{n+\mu} = \frac{f(n+\mu+1)}{\mu!\,f(0)}(1+x)^\mu \log(1+x) + D^n. 0, \tag{25}$$

the values of the letters being the same as in (8) and (10) respectively.

6. In Peacock's case, using formula (23) and writing $m = n = \frac{1}{2}$, we have

$x < 1$

$$\begin{aligned} D^{\frac{1}{2}}(1+x)^{\frac{1}{2}} &= \frac{x^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} + \frac{\frac{1}{2}x^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} + \frac{\frac{1}{2}(-\frac{1}{2})x^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})x^{\frac{5}{2}}}{\Gamma(\frac{7}{2})} + \dots \\ &= \pi^{-\frac{1}{2}}\left(x^{-\frac{1}{2}} + x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3} + \frac{x^{\frac{5}{2}}}{5} - \frac{x^{\frac{7}{2}}}{7} + \dots\right) \\ &= \pi^{-\frac{1}{2}}(x^{-\frac{1}{2}} + \tan^{-1}x^{\frac{1}{2}}); \end{aligned} \tag{26}$$

and if to verify we semi-differentiate again

$$\begin{aligned} D(1+x)^{\frac{1}{2}} &= \frac{x^{-1}}{\Gamma(0)} + \frac{\frac{1}{2}x^0}{\Gamma(1)} + \frac{\frac{1}{2}(-\frac{1}{2})x}{\Gamma(2)} + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})x^2}{\Gamma(3)} + \dots \\ &= \tfrac{1}{2}\left\{1 - \tfrac{1}{2}x + (-\tfrac{1}{2})(-\tfrac{3}{2})\frac{x^2}{2!} + \dots\right\} = \tfrac{1}{2}(1+x)^{-\frac{1}{2}}, \end{aligned}$$

no complementary function is here needed.

But using (22) in both Peacock's and Liouville's cases, the complementary function is required to correct our results; in fact, using the expanded form of (22), viz.

$x > 1$

$$D^n(1+x)^m = f(m+1)\left\{\frac{x^{m-n}}{f(m-n+1)} + \frac{x^{m-n-1}}{f(m-n)} + \frac{x^{m-n-2}}{2\,!\,f(m-n-1)} + \dots\right\},$$

and putting $m = n = \frac{1}{2}$, this is

$$D^{\frac{1}{2}}(1+x)^{\frac{1}{2}} = f(\tfrac{3}{2})\left\{\frac{x^0}{f(1)} + \frac{x^{-1}}{f(0)} + \frac{x^{-2}}{2\,!\,f(-1)} + \frac{x^{-3}}{3\,!\,f(-2)} + \dots\right\}$$

$= \Gamma(\frac{3}{2}) + 0$ in Peacock's case, and a complementary function is required, on semi-differentiating again, to supply the missing terms. In Liouville's case, on the other hand, none of the terms disappear, but the first term is infinite since $f(1) = 0$ and must be replaced by a logarithm by borrowing from the complementary function, as in Art. 2, so that in this case we have, after reducing,

$$D^{\frac{1}{2}}(1+x)^{\frac{1}{2}} = \frac{f(\frac{3}{2})}{f(0)}\left\{\log\psi + x^{-1} - \frac{x^{-2}}{2} + \frac{x^{-3}}{3} - \frac{x^{-4}}{4} + \dots\right\}$$

$$= \frac{f(\frac{3}{2})}{f(0)}\log(1+x), \tag{27}$$

which agrees with formula (25).

For a further verification of (27) it may be noticed that when $m = \frac{1}{2}$ and $n = \frac{3}{2}$ in (22), omitting the complementary function we have

$$D^{\frac{3}{2}}(1+x)^{\frac{1}{2}} = \frac{f(\frac{3}{2})}{f(0)}(1+x)^{-1},$$

which may also be deduced by differentiating (27).

7. From (26) it is easy to obtain

$$D^{\frac{1}{2}}\tan^{-1}x^{\frac{1}{2}} = \Gamma(\tfrac{3}{2})(1+x)^{-\frac{1}{2}}, \tag{28}$$

Peacock's formula being used and the complementary function omitted; and this is a particular case of a more general theorem, for if

$$\phi(x) = A_1x + \frac{A_3x^3}{3\,!} + \frac{A_5x^5}{5\,!} + \frac{A_7x^7}{7\,!} + \dots$$

$$D^{\frac{1}{2}}\phi(x^{\frac{1}{2}}) = D^{\frac{1}{2}}\left(A_1x^{\frac{1}{2}} + \frac{A_3x^{\frac{3}{2}}}{3\,!} + \frac{A_5x^{\frac{5}{2}}}{5\,!} + \dots\right)$$

$$= \Gamma(\tfrac{3}{2})\left\{A_1 + \frac{A_3 3\,x}{2\,.\,3\,!\,\Gamma(2)} + \frac{A_5 5\,.\,3\,x^2}{2^2.\,5\,!\,\Gamma(3)} + \frac{A_7 7\,.\,5\,.\,3\,x^3}{2^3.\,7\,!\,\Gamma(4)} + \dots\right\}$$

$$= \Gamma(\tfrac{3}{2})\left(A_1 + \frac{A_3x}{2^2} + \frac{A_5x^2}{2^2.\,4^2} + \frac{A_7x^3}{2^2.\,4^2.\,6^2} + \dots\right), \tag{29}$$

and in particular, if $A_1 = 1$, $A_3 = 1^2$, $A_5 = 1^2.\,3^2$, $A_7 = 1^2.\,3^2.\,5^2$, ... (29) becomes

$$D^{\frac{1}{2}}\sin^{-1}x^{\frac{1}{2}} = \Gamma(\tfrac{3}{2})\left(1 + \frac{1^2}{2^2}x + \frac{1^2.\,3^2}{2^2.\,4^2}x^2 + \frac{1^2.\,3^2.\,5^2}{2^2.\,4^2.\,6^2}x^3 + \dots\right)$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)}{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - x \sin^2\theta}} = \pi^{-\frac{1}{2}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - x \sin^2\theta}},$$

and a similar expression can be found for $\int_0^{\frac{\pi}{2}} \sqrt{1 - x \sin^2\theta}\, d\theta$, viz. in this case $\phi(x) = \int_0^x (1 - x^2)^{\frac{1}{2}}\, dx$, while in the former case $\phi(x) = \int_0^x \frac{dx}{\sqrt{1 - x^2}}$, and throughout $x < 1$.

8. A proof of Leibnitz's Theorem, which does not assume $D^n e^{ax} = a^n e^{ax}$, will conclude the present paper.

Let u, v be two functions of x capable of expansion in powers of x, and suppose that

$$u = \Sigma\, A_m x^m, \qquad v = \Sigma\, B_n x^n; \qquad \text{then } uv = \sum A_m B_n x^{m+n}$$

[this last summation extending to all values of m which occur in u, and of n in v],

Hence $$D^\mu uv = D^\mu . 0 + \sum A_m B_n D^\mu x^{m+n}$$

$$= D^\mu . 0 + \sum A_m B_n \frac{f(m + n + 1)}{f(m + n - \mu + 1)} x^{m+n-\mu}. \tag{30}$$

Now, omitting all complementary functions, it will be shown that the series

$$D^\mu u . v + \mu D^{\mu-1} u . v' + \frac{\mu(\mu - 1)}{1.2} D^{\mu-2} u . v'' + \dots .$$

is equal to the second term of (30); for the series is

$$\sum A_m B_n \left\{ \frac{f(m + 1)}{f(m - \mu + 1)} x^{m-\mu} x^n + \mu \frac{f(m + 1)}{f(m - \mu + 2)} x^{m-\mu+1} n x^{n-1} \right.$$
$$\left. + \frac{\mu(\mu - 1)}{1.2} \cdot \frac{f(m + 1)}{f(m - \mu + 3)} x^{m-\mu+2} n(n - 1) x^{n-2} + \dots . \right\},$$

which, by (3),

$$= \sum A_m B_n \frac{C_m}{C_{m-\mu}} \frac{x^{m+n-\mu}}{\Gamma(-\mu)} \left\{ \frac{\Gamma(m + 1)\,\Gamma(-\mu)}{\Gamma(m - \mu + 1)} - n \frac{\Gamma(m + 1)\,\Gamma(1 - \mu)}{\Gamma(m - \mu + 1)} \right.$$
$$\left. + \frac{n(n - 1)}{1.2} \frac{\Gamma(m + 1)\,\Gamma(2 - \mu)}{\Gamma(m - \mu + 3)} - \frac{n(n - 1)(n - 2)}{1.2.3} \frac{\Gamma(m + 1)\,\Gamma(3 - \mu)}{\Gamma(m - \mu + 4)} + \dots . \right\}$$

$$= \sum A_m B_n \frac{C_m}{C_{m-\mu}} \frac{x^{m+n-\mu}}{\Gamma(-\mu)} \int_0^1 (1 - x)^m x^{-\mu-1} \left(1 - nx + \frac{n . n - 1}{1.2} x^2 \dots .\right) dx$$

$$= \sum A_m B_n \frac{C_m}{C_{m-\mu}} \frac{x^{m+n-\mu}}{\Gamma(-\mu)} \cdot \frac{\Gamma(m + n + 1)\,\Gamma(-\mu)}{\Gamma(m + n - \mu + 1)};$$

and if v only contains integral powers of x, n is integral and $C_m = C_{m+n+1}$; so that this expression reduces to the same form as (30).

Hence, if u be capable of expansion in any series of powers of x, and v in a series of integral powers,

$$D^\mu uv = D^\mu . 0 + \left(D^\mu u . v + \mu D^{\mu-1} u . v' + \frac{\mu(\mu-1)}{1.2} D^{\mu-2} u . v'' + \dots\right).$$

The restriction of the nature of the expansion of v to *positive integral* powers of x is attended with this advantage that the coefficient of $x^{m+n-\mu}$ in the series for $D^\mu uv$ always consists of a finite number of terms, viz. $n+1$.

Note on a Theorem for Expanding Functions of Functions.

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IT has just come to my knowledge that the "theorem for expanding functions of functions" lately published by me (Vol. II. p. 348) was essentially anticipated, twenty years earlier, by Mr. S. Roberts, F.R.S. Mr. Roberts's brief, yet sufficient statement (Quarterly Journal, Vol. IV. p. 195) is as follows, putting $\phi x = a_0 + a_1 x + a_2 x^2 + \dots$, and $\Pi = a_1 \frac{d}{da_0} + 2\,a_2 \frac{d}{da_1} + 3\,a_3 \frac{d}{da_2} + \dots$:—

"It will be observed that, since $\Pi^n \phi x = D_x^n \phi x$, the notation of (1) applies generally to a function of ϕx and the differentials of ϕx, and we may write

$$F(\phi, \phi', \phi'', \dots) = \epsilon^{x\Pi} F(a_0, a_1, 2\,a_2, \dots)."$$

MILWAUKEE, Aug. 3, 1880.